

# Weak measurement-based state estimation of Gaussian states of one-variable quantum systems

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We present a scheme to estimate Gaussian states of one-dimensional continuous variable systems, based on weak (unsharp) quantum measurements. The estimation of a Gaussian state requires us to find position ( $q$ ), momentum ( $p$ ) and their second order moments. We measure  $q$  weakly and follow it up with a projective measurement of  $p$  on half of the ensemble, and on the other half we measure  $p$  weakly followed by a projective measurement of  $q$ . In each case we use the state twice before discarding it. We compare our results with projective measurements and demonstrate that under certain conditions such weak measurement-based estimation schemes, where recycling of the states is possible, can outperform projective measurement-based state estimation schemes.

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## I. INTRODUCTION

State determination for a physical system hinges on being able to non-invasively measure its relevant parameters. However, a measurement performed on a quantum system is by definition invasive, and hence quantum state estimation relies on measurements made on an ensemble of identically prepared systems. Quantum state estimation process is thus intrinsically statistical in nature, with its accompanying ambiguities and uncertainties. [1, 2]. There is no direct measurement possible for the quantum state of a single system and for the estimation of a state, we are required to determine the expectation values of a set of incompatible observables. The accuracy of such a determination depends upon the size of the ensemble and ideally we need an infinite size ensemble to obtain the precise values of these expectations. However, if we are given a small ensemble with a fixed number of identically prepared states, how can we effectively extract information from it? The wave function collapse associated with projective measurements limits us from using each member of the ensemble more than once. The problem of state estimation has been investigated by physicists since the inception of quantum information theory [3]. Apart from the direct use of projective measurements, there exist other methods of state tomography which try to extract information from a system in different ways. Some prescriptions use the information gained about the system from one measurement to decide on the next measurement [4]. Others employ maximum likelihood technique and numerical optimization [5], or use repeated weak measurements on a single system [6], to maximize the information gain. Weak or unsharp measurements potentially hold promise for state estimation because of their non-invasiveness which allows state recycling. On the one hand the disturbance caused by a

weak measurement is less, however on the other hand it also gives us limited information. The challenge therefore is to find a balance i.e. an intermediate regime of weakness, which leads to optimal information gain. The idea has been explored recently in the context of a qubit with a small ensemble size [7, 8]. Weak measurements coupled with postselection have also been employed in the problem of state estimation where a complete characterization of the postselected quantum statistics [9] or the direct measurement of the quantum wavefunction is used [10–12]. State measurement schemes based on weak measurement tomography have also been recently proposed [13, 14]. There have also been critical analysis of these schemes [15].

For continuous variable systems, quasi-probability distributions including the Wigner distributions can be tomographed by measuring the rotated quadrature components [16]. There are homodyne and heterodyne schemes for estimation of squeezed Gaussian states [17] and squeezed thermal Gaussian states [18, 19]. Schemes using a single photon detector instead of a homodyne detector to characterize Gaussian states have been proposed [20]. The possible advantage offered by an entangled Gaussian probe to estimate the displacement of a continuous variable state has also been explored [21]. The importance of maximum likelihood methods has been emphasized [22, 23] and state reconstruction has been described [24, 25]. In a different direction, Arthurs and Kelly aimed to simultaneously measure position and momentum of a general wavefunction by coupling two different apparatuses with the system [26]. Since  $q$  and  $p$  are noncommuting observables, this leads to an unsharp measurement. Symplectic tomography has been used to estimate the master equation parameters in an open setting for a single mode system [27].

Alternatively, one can reuse each member of the ensemble if the first measurement is done weakly enough such that the disturbance induced is very small [7, 28]. A similar idea has been utilized, albeit in a different direction, in the construction of loophole free hybrid Bell-Leggett-

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Garg inequalities [29, 30]. Weak or unsharp measurements are performed by weakly coupling the device to the quantum system [7, 31–33]. Although the noise produced in such measurements is small, which should serve our purpose well, the information obtained is also very low. Therefore, there is a tradeoff between the disturbance and information gain. To effectively use weak measurements for state estimation, we need to optimize the process.

In this work we restrict ourselves to the realm of a special class of states of one continuous variable quantum systems called Gaussian states. These are states with Gaussian-Wigner quasiprobability distributions and include coherent states, squeezed states and thermal squeezed states. The Gaussian states are determined by the first and second moments of position and momentum. We explore the advantage of the scheme involving weak measurements in estimating the Gaussian states over projective measurements. We show that with an optimal strength of the weak measurement, our technique is more powerful in the determination of the Wigner quasiprobability distribution when the ensemble size available is small. We have chosen the meter state in the form of a minimum uncertainty squeezed Gaussian state and the tuning of the weakness of a measurement is achieved by changing the squeezing of the position quadrature. This scheme is tested for average performance over a large number of states and over a large number of runs to kill statistical fluctuations. We also take Gaussian states at different temperatures and check whether the efficacy of our method depends in any way on temperature.

The paper is arranged as follows: In section II we collate all the background material necessary for the problem. We describe continuous variable states and Wigner's quasiprobability distributions in brief. Symplectic methods are described briefly. Section III gives a description of weak measurement in quantum mechanics when applied to Gaussian states. In section IV we describe how to perform tomography of Gaussian states using our method. We provide conclusions in section V.

## II. GAUSSIAN STATES

Let us consider a one-dimensional quantum system with the position and momentum operators  $\hat{q}$  and  $\hat{p}$ , satisfying the commutation relation

$$[\hat{q}, \hat{p}] = i \quad (\hbar = 1) \quad (1)$$

The corresponding eigenkets are  $|q\rangle$  and  $|p\rangle$  for a complete set [1], and are defined as

$$\begin{aligned} \hat{q}|q\rangle &= q|q\rangle, \quad \hat{p}|p\rangle = p|p\rangle \\ \langle q|q'\rangle &= \delta(q - q'), \quad \langle p|p'\rangle = \delta(p - p') \text{ and } \langle q|p\rangle = e^{ipq} \end{aligned} \quad (2)$$

An arbitrary mixed density operator in position and momentum bases can be represented by

$$\begin{aligned} \hat{\rho}(q, q') &= \int dq dq' f(q, q') |q\rangle \langle q'| \\ \hat{\rho}(p, p') &= \int dp dp' \tilde{f}(p, p') |p\rangle \langle p'| \end{aligned} \quad (3)$$

where  $f(q, q')$  is a function of real variables  $q$  and  $q'$  while  $\tilde{f}(p, p')$  is a function of the real variables  $p$  and  $p'$ .

An alternative and equivalent way of representing the system state [34] is via its Wigner distribution  $W(q, p)$  given by

$$W(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle e^{ipy} \quad (4)$$

The probability distributions corresponding to position and momentum can be obtained by computing the marginals

$$\begin{aligned} P(q) &= \int_{-\infty}^{\infty} W(q, p) dp \\ P(p) &= \int_{-\infty}^{\infty} W(q, p) dq \end{aligned} \quad (5)$$

and can be used to calculate expectation values of arbitrary observables via the symmetric ordering rule [34, 35].

The second order moments of position and momentum corresponding to a quantum state are given by

$$\begin{aligned} (\Delta q)^2 &= \langle q^2 \rangle - \langle q \rangle^2, \quad (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \\ \Delta(q, p) &= \frac{1}{2} \langle \{\hat{q} - \langle \hat{q} \rangle, \hat{p} - \langle \hat{p} \rangle\} \rangle \end{aligned} \quad (6)$$

and they obey the Schrödinger uncertainty principle given by

$$(\Delta q)^2 (\Delta p)^2 \geq \left| \frac{1}{2} \langle \{\hat{q}, \hat{p}\} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle \right|^2 + \left| \frac{1}{2i} \langle [\hat{q}, \hat{p}] \rangle \right|^2 \quad (7)$$

A compact way to represent the second order moments is via the variance matrix  $V$  given by

$$V = \begin{pmatrix} (\Delta q)^2 & \Delta(q, p) \\ \Delta(q, p) & (\Delta p)^2 \end{pmatrix} \quad (8)$$

and the uncertainty condition re-expressed in terms of the variance matrix takes the elegant form [35, 36]

$$\begin{aligned} V + \frac{i}{2} \beta &\equiv \text{positive semidefinite} \\ \beta &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta^{-1} = \beta^T = -\beta, \quad \text{Det}(\beta) = 1 \end{aligned} \quad (9)$$

The subclass of states for which the Wigner distribution is a Gaussian function are called Gaussian states and play an important role in quantum optics and quantum information [37–39]. All states with Gaussian wave functions are Gaussian, however the class of Gaussian states is a much bigger class and includes mixed states. The Wigner

representation corresponding to a general Gaussian state centered at the origin of the phase space is given by,

$$W(\xi) = \frac{1}{\pi} \sqrt{|G|} e^{-\xi^T G \xi} \quad (10)$$

where

$$\xi = \begin{pmatrix} q \\ p \end{pmatrix} \quad G = G^* = G^T \quad (11)$$

The matrix  $G$  is related to the variance matrix  $V$  as

$$V = \frac{1}{2} G^{-1} \quad (12)$$

If the center of the Gaussian is located at a point  $(q_0, p_0)$ , this can be achieved by action of a displacement operator  $\hat{D}(q_0, p_0)$  which acts on canonical operators as

$$\hat{\xi} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \rightarrow \hat{D}(q_0, p_0) \hat{\xi} \hat{D}(q_0, p_0)^{-1} = \begin{pmatrix} \hat{q} - q_0 \\ \hat{p} - p_0 \end{pmatrix} = \hat{\xi} \quad \hat{D}(q_0, p_0) = e^{i(p_0 \hat{q} - q_0 \hat{p})} \quad (13)$$

and leads to a point transformation of the Wigner function. For the Gaussian-Wigner function this amounts to shifting the center to the location  $(q_0, p_0)$ . The matrix  $G$  can be written as:

$$G = \hat{U} S^T G_0 S \hat{U}^{-1} \quad (14)$$

where  $S$  is a diagonal symplectic matrix belonging to the group  $Sp(2, R)$ ,  $\hat{U}$  is a rotation matrix and  $G_0$  is proportional to identity. A total of three real parameters are involved in describing  $G$ . We further restrict ourselves to a special class of Gaussians where  $G$  is described by two parameters, namely, the temperature and squeezing [35, 36]. Setting

$$\hat{U} = I, \quad S = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^u \end{pmatrix}, \quad G_0 = \kappa I \quad (15)$$

with  $\kappa = \tanh\left(\frac{\omega}{2kT}\right)$  representing the temperature  $T$ ,  $k$  being the Boltzmann's constant and  $\omega$  having the units of frequency.

The corresponding variance matrix in this case is diagonal

$$V = \begin{pmatrix} (\Delta q)^2 & 0 \\ 0 & (\Delta p)^2 \end{pmatrix} \quad (16)$$

At  $T = 0K$  or  $\kappa = 1$  if we have  $\Delta q = \Delta p = \frac{1}{\sqrt{2}}$ , the Gaussian state is a coherent state. A coherent state is represented by a circle of radius  $\frac{1}{\sqrt{2}}$  in phase space. Now at  $T = 0K$ , if  $\Delta q$  and  $\Delta p$  happen to be unequal, the state is a squeezed state. Such a Gaussian state is represented by an ellipse. The center of a general coherent or a general squeezed state may not be at the origin of phase space and in such cases it is said to be a displaced

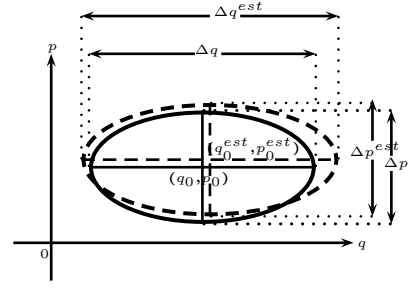


FIG. 1. The Gaussian state, represented by an ellipse, in the phase space. The actual state is an ellipse bounded by continuous line, centered at  $(q_0, p_0)$  and with spreads  $\Delta q$  and  $\Delta p$ . The estimated state is represented by another ellipse, bounded by a broken line, centered at  $(q_0^{est}, p_0^{est})$  and has spreads  $\Delta q^{est}$  and  $\Delta p^{est}$ .

coherent or a displaced squeezed state. The displacement of a Wigner function  $W(\xi)$ , centered at the origin is achieved by means of a displacement operator  $\hat{D}(q_0, p_0)$  which takes the center to  $(q_0, p_0)$ . For a non-zero temperature, the value of the product  $\Delta q \Delta p$  is greater than  $\frac{1}{2}$  (for coherent states it is equal to half) and it increases with a rise of temperature. Pictorially, the class of states can be represented by an ellipse in phase space with center at  $(q_0, p_0)$  and semimajor axis oriented along either  $q$  or  $p$  and is depicted in Figure 1.

### III. WEAK MEASUREMENTS ON GAUSSIAN STATES

Consider a system represented by a displaced Gaussian-Wigner distribution function represented by  $W_s(\xi_s)$  and characterized by two real displacement parameters: one temperature parameter and one squeezing parameter. Such a Wigner function can be obtained by starting with the centered Gaussian-Wigner function given in Equation (10) with parameters chosen as per Equations (14) and (15) and applying a displacement operator  $\hat{D}(q_0, p_0)$  as described in the previous section. Given that the system is in such a state, our goal is to estimate the state.

For the purposes of measurement, consider a meter which is a macroscopic pointer with position and momentum variables  $q_{m_1}$  and  $p_{m_1}$ . The meter is chosen to be in a squeezed coherent state represented by a Wigner distribution  $W(\xi_{m_1})$  such that  $\Delta q_{m_1} \Delta p_{m_1} = \frac{1}{2}$  and  $\Delta q_{m_1} \neq \Delta p_{m_1}$ . Here we take temperature to be zero. As we shall see, when we employ this meter to measure the position, the strength of the measurement can be varied by changing the squeezing of the meter along the position quadrature. The larger values of the squeezing parameter correspond to a stronger measurement while the smaller values of the squeezing parameter correspond to a weaker measurement. Similarly, if we are measuring momentum one can tune the measurement strength by

varying the value of squeezing of the variable  $q_{m_2}$ .

The system and the meter form a composite system and the joint Wigner function representing this two degrees of freedom system can be obtained by multiplying the two individual Wigner functions. For such a system it is natural to define a four dimensional column vector of phase space variables as

$$\Xi = \begin{pmatrix} q \\ q_{m_1} \\ p \\ p_{m_1} \end{pmatrix}. \quad (17)$$

The phase space displacement of the system variables given in Equation (13) acts on this column vector to give us a displaced vector

$$\tilde{\Xi} = \begin{pmatrix} q - q_0 \\ q_{m_1} \\ p - p_0 \\ p_{m_1} \end{pmatrix}. \quad (18)$$

In terms of the above column vector The joint Wigner function becomes

$$W(\Xi) = \frac{1}{\pi^2} \sqrt{|G|} e^{-\tilde{\Xi}^T G \tilde{\Xi}} \quad (19)$$

The matrix  $G$  is a diagonal  $4 \times 4$  matrix

$$G = \text{Diag}((\Delta q)^2, (\Delta q_{m_1})^2, (\Delta p)^2, (\Delta p_{m_1})^2) \quad (20)$$

$$W(\Xi) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(q-q_0)^2}{(\Delta q)^2} + \frac{(p-p_0)^2}{(\Delta p)^2} + \frac{q_{m_1}^2}{(\Delta q_{m_1})^2} + \frac{p_{m_1}^2}{(\Delta p_{m_1})^2} \right) \right]}{4\pi^2 \Delta q \Delta p \Delta q_{m_1} \Delta p_{m_1}} \quad (21)$$

When we perform a measurement (weak or strong) of the position  $q$ , we switch on the following interaction Hamiltonian between the system and the meter degrees of freedom

$$\hat{H} = \delta(t - t_1) \hat{q} \hat{p}_{m_1} \quad (22)$$

The corresponding unitary transformation on the composite system-meter is

$$\hat{U} = e^{-i \int \hat{H} dt} \quad (23)$$

In the language of the Wigner quasi-probability distribution a unitary operation  $\hat{U}$  is equivalent to a symplectic transformation  $\mathcal{S}$

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (24)$$

satisfying

$$\mathcal{S}^T \beta_2 \mathcal{S} = \mathcal{S} \beta_2 \mathcal{S}^T = \beta_2 \text{ with } \beta_2 = \begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -I_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}. \quad (25)$$

The symplectic transformation corresponding to the interaction Hamiltonian acts on the phase variables by multiplication

$$\Xi' = \mathcal{S} \Xi \quad (26)$$

leading to the computation of the transformed Wigner function under this symplectic transformation

$$W'(\Xi) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(q-q_0)^2}{(\Delta q)^2} + \frac{(p+p_{m_1}-p_0)^2}{(\Delta p)^2} + \frac{(q_{m_1}-q)^2}{(\Delta q_{m_1})^2} + \frac{p_{m_1}^2}{(\Delta p_{m_1})^2} \right) \right]}{4\pi^2 \Delta q \Delta p \Delta q_{m_1} \Delta p_{m_1}}. \quad (27)$$

The Wigner function of the meter after the above interaction is obtained by integrating over the system variables  $q$  and  $p$  and is given by

$$W'_{m_1}(\xi_{m_1}) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(q_{m_1}-q_0)^2}{(\Delta q_{m_1})^2 + (\Delta q)^2} + \frac{p_{m_1}^2}{(\Delta p_{m_1})^2} \right) \right]}{2\pi \Delta q_{m_1} \Delta p_{m_1} \Delta q \sqrt{\frac{1}{(\Delta q_{m_1})^2} + \frac{1}{(\Delta q)^2}}}. \quad (28)$$

We can see at this point that the state of the meter has become correlated with the state of the system. However, as the meter is a macroscopic entity, its very observation leads to the collapse of its wavefunction and gives us a definite value. Thus the probability density for the meter to show a reading  $q_{m_1}$

$$P(q_{m_1}) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(q_{m_1}-q_0)^2}{(\Delta q_{m_1})^2 + (\Delta q)^2} \right) \right]}{\sqrt{2\pi} \Delta q_{m_1} \Delta q \sqrt{\frac{1}{(\Delta q_{m_1})^2} + \frac{1}{(\Delta q)^2}}}. \quad (29)$$

On the other hand, the reduced state of the system after the measurement interaction represented by the symplectic transformation is obtained by integrating over the meter degrees of freedom leading to the Wigner function for the system

$$W'_s(\xi_s) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(p-p_0)^2}{(\Delta p_{m_1})^2 + (\Delta p)^2} + \frac{(q-q_0)^2}{(\Delta q)^2} \right) \right]}{2\pi \Delta p_{m_1} \Delta p \Delta q \sqrt{\frac{1}{(\Delta p_{m_1})^2} + \frac{1}{(\Delta p)^2}}} \quad (30)$$

In the weak measurement limit  $\Delta q_{m_1}$  is large (i.e. the initial meter state is prepared in distributions wide in position). Since we have chosen the meter to be in a squeezed coherent state, this corresponds to a high degree of squeezing in the momentum quadrature of the initial meter state. In this limit we have

$$\Delta p_{m_1} \rightarrow 0 \Rightarrow W'_s \rightarrow W_s. \quad (31)$$

Hence, weak measurement causes controllable disturbance to the state and the disturbance vanishes in the limit of extremely weak measurement. However, if we make the measurement too weak, the correlation between the meter state and the system state diminishes. In the limit of extremely weak measurement, where no disturbance is caused, we do not learn anything about the system from observing the meter.

In our scheme, the first measurement that we perform is a weak measurement of position  $q$  with a tunable strength as described above. Subsequently, we carry out a projective measurement of momentum  $p$  on this system, then the probability density for obtaining any momentum as obtained from the modified system Wigner function given in Equation (30) is given by,

$$P(p) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(p-p_0)^2}{(\Delta p_{m_1})^2 + (\Delta p)^2} \right) \right]}{2\pi \Delta p_{m_1} \Delta p \sqrt{\frac{1}{(\Delta p_{m_1})^2} + \frac{1}{(\Delta p)^2}}} \quad (32)$$

In the reverse scenario where we do a weak measurement of momentum  $p$  followed by a projective measurement of position  $q$ , the composite system-meter system Wigner function after the measurement interaction given by the Hamiltonian

$$\hat{H}' = \delta(t - t'_1) \hat{p} \hat{p}_{m_2} \quad (33)$$

is given by

$$W''(\Xi) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(p-p_0)^2}{(\Delta p)^2} + \frac{p_{m_2}^2}{(\Delta p_{m_2})^2} + \frac{(q-p_{m_2}-q_0)^2}{(\Delta q)^2} + \frac{(p-q_{m_2})^2}{(\Delta q_{m_2})^2} \right) \right]}{4\pi^2 \Delta p_{m_2} \Delta p \Delta q_{m_2} \Delta q} \quad (34)$$

where  $q_{m_2}$  and  $p_{m_2}$  denote the position and momentum coordinates of the meter measuring momentum  $p$  of the system. The Wigner of the meter alone is given by

$$W''_{m_2}(\xi_{m_2}) = \frac{\exp \left[ -\frac{(q_{m_2}-p_0)^2}{2((\Delta p)^2 + (\Delta q_{m_2})^2)} - \frac{p_{m_2}^2}{2(\Delta p_{m_2})^2} \right]}{2\pi \Delta p_{m_2} \Delta p \Delta q_{m_2} \sqrt{\frac{1}{(\Delta p)^2} + \frac{1}{(\Delta q_{m_2})^2}}} \quad (35)$$

giving the probability density of the meter to show a reading  $q_{m_2}$  being

$$P(q_{m_2}) = \frac{\exp \left[ -\frac{(q_{m_2}-p_0)^2}{2((\Delta p)^2 + (\Delta q_{m_2})^2)} \right]}{\sqrt{2\pi} \Delta p \Delta q_{m_2} \sqrt{\frac{1}{(\Delta p)^2} + \frac{1}{(\Delta q_{m_2})^2}}} \quad (36)$$

The corresponding system Wigner function becomes

$$W''_s(\xi_s) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(p-p_0)^2}{(\Delta p_{m_2})^2 + (\Delta q)^2} + \frac{(q-q_0)^2}{(\Delta p_{m_2})^2 + (\Delta q)^2} \right) \right]}{2\pi \Delta p_{m_2} \Delta p \Delta q \sqrt{\frac{1}{(\Delta p_{m_2})^2} + \frac{1}{(\Delta q)^2}}} \quad (37)$$

As before, in the weak measurement limit the disturbance caused in the system is limited and we have

$$\Delta q_{m_2} \rightarrow 0 \Rightarrow W''_s \rightarrow W_s. \quad (38)$$

On this state we perform a projective measurement of position  $q$  giving us the probability density for getting a result  $q$

$$P(q) = \frac{\exp \left[ -\frac{1}{2} \left( \frac{(q-q_0)^2}{(\Delta p_{m_2})^2 + (\Delta q)^2} \right) \right]}{\sqrt{2\pi} \Delta p_{m_2} \Delta q \sqrt{\frac{1}{(\Delta p_{m_2})^2} + \frac{1}{(\Delta q)^2}}}. \quad (39)$$

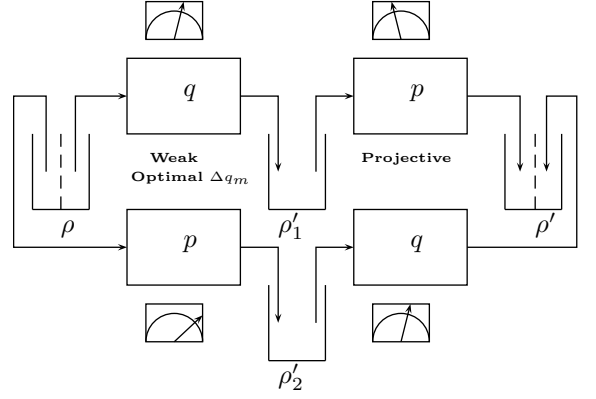


FIG. 2. The algorithm to implement our scheme where we first divide the ensemble  $\rho$  into two parts. For one half of the ensemble we measure position  $q$  weakly (the weakness being defined by the initial spread in position  $\Delta q_{m_1}$  of the meter) leading to a disturbed ensemble  $\rho'_1$ . On every member of the ensemble  $\rho'_1$  we carry out a projective measurement of momentum  $p$ . With the other half of the initial ensemble  $\rho$ , momentum  $p$  is measured weakly leading to a disturbed ensemble  $\rho'_2$  on which a projective measurement of  $q$  is carried out.

#### IV. ESTIMATION OF GAUSSIAN STATES USING WEAK MEASUREMENTS

##### A. The prescription

In order to perform complete state tomography of any Gaussian state of the form discussed earlier, we are required to estimate the center of the Gaussian Wigner function,  $(q_0, p_0)$  and the spreads  $\Delta q$  and  $\Delta p$ . Hence it is necessary to measure both position  $q$  and momentum  $p$  of the system as accurately as possible. To this end, we divide the initial ensemble of identically prepared systems into two equal parts. On every member of one part we perform a weak measurement of position  $q$ . The strength of the measurement is governed by the initial squeezing of the position quadrature of the meter determining the initial variance  $\Delta q_{m_1}$  of the meter state. The larger the value of  $\Delta q_{m_1}$ , weaker is the measurement strength and vice versa. The meter reading is recorded in each case and the final states of all the members are collected to generate a second ensemble. The members of this ensemble are now used as the initial states of a second measurement, which is a projective measurement of momentum  $p$ . As before the meter readings of this measurement are noted. Now the process is repeated with the members of the second part of the initial ensemble where we first measure  $p$  weakly, with the strength of the measurement determined through  $\Delta q_{m_2}$  and then carry out a projective measurement of  $q$ . In all further analysis and discussions we take  $\Delta q_{m_1} = \Delta q_{m_2} = \Delta q_m$ . A summary of the procedure is illustrated in Figure 2. The entire algorithm is repeated over many runs to rule out statistical fluctuations. It is worth noting that the initial squeezing

of the relevant quadrature which determines the strength of the measurement is a tunable parameter in our hand. Although we call certain measurements “weak”, we actually mean that it is not too strong to be projective and not too feeble to induce large errors to the measurement outcomes. The main point is that the measurements are weak enough and do not cause the complete collapse of the state so that it can be used for subsequent measurements. The expectation values obtained from the  $q$  and  $p$  measurements are used to estimate the values of  $(q_0, p_0)$ , and the spreads  $\Delta q$  and  $\Delta p$ .

Looking at the Equations (28) and (34) reveals that the information about the system has flowed into the meter. In fact the meter is now centered over  $(q_0, p_0)$  which is the center of the initial system state. We carry out simulations using the meter reading probabilities given in Equations (29), (39), (36) and (39). We take different ensemble sizes of member numbers 6, 8, 10 and 20 respectively with randomly generated Gaussian states. Each virtual experiment is repeated over 1000 runs. The quantities  $q_0$  and  $p_0$  for a state are estimated by taking the mean over the  $q$  and  $p$  measurements while  $\Delta q$  and  $\Delta p$  are estimated from the corresponding variances. The order of measurement of  $q$  and  $p$  is reversed for the second part of the ensemble to rule out the possibility of preferential treatment of any of the observables.

In the scheme involving projective measurements only, we divide the original ensemble into two parts and perform  $q$  and  $p$  measurements independently on the individual members of these parts. No sequential measurements are possible here because of the wavefunction collapses after the measurement.

The accuracy of the state estimate is measured via the following distance measures

$$\begin{aligned} d_1 &= (q_0 - q_0^{est})^2 + (p_0 - p_0^{est})^2 \\ d_2 &= (\Delta q - \Delta q^{est})^2 + (\Delta p - \Delta p^{est})^2 \end{aligned} \quad (40)$$

where  $q_0^{est}$ ,  $p_0^{est}$ ,  $\Delta q^{est}$  and  $\Delta p^{est}$  are the estimated values of  $q_0$ ,  $p_0$ ,  $\Delta q$  and  $\Delta p$ , respectively. The parameter  $d_1$  is a measure of how well our method is able to estimate the center of the Gaussian and  $d_2$  gives a measure of how well the spreads of the Gaussian have been estimated. The two measures  $d_1$  and  $d_2$  represent closeness in position and width of the estimated Wigner distribution from the original Wigner distribution respectively. We can immediately see that the lower these distances, the better the estimates. For a perfect estimate the values should go to zero.

## B. Performance of the scheme

To study the average performance of our scheme for squeezed displaced thermal states, we begin by numerically generating 100 Gaussian states at a particular temperature, with randomly chosen values of displacement and squeezing. To generate these states, the value of the

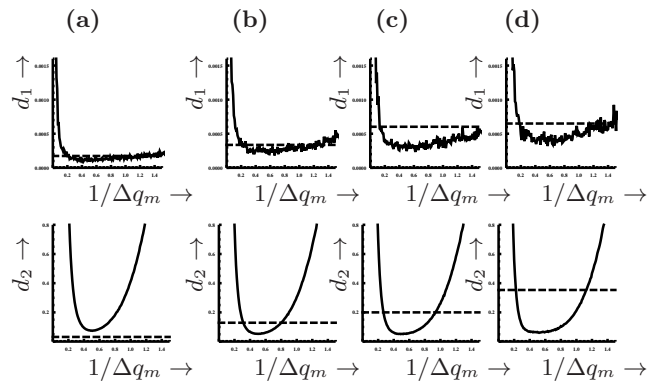


FIG. 3. The efficacy of our method as compared to projective measurements for  $\kappa = 1$  using averages over 100 random states further averaged over 1000 runs. The behaviors of  $d_1$  and  $d_2$  are plotted with  $1/\Delta q_m$  for ensemble sizes (a) 20 (b) 10 (c) 8 and (d) 6. The corresponding projective measurement results are plotted as dotted lines. While the method performs well in estimating the position of the Gaussian states for all ensemble sizes, as represented by  $d_1$ , it provides a clear advantage for estimating the spreads represented by  $d_2$  over projective measurements in the case of a small ensemble of size 6.

parameter  $u$  in Equation (15) is varied between  $-1$  and  $+1$  according to a uniform distribution. Similarly, the centers of the Gaussians are also chosen randomly using uniform distributions between  $-3$  and  $+3$  for both  $q_0$  and  $p_0$ .

With each of these 100 random states, we numerically carry out the prescription given in subsection IV A on a fixed number of identical copies of the state determining the ensemble size. The simulation is carried out with the help of the results obtained in section III. The distance measures  $d_1$  and  $d_2$  used to compare the efficacy of our method with projective measurements are computed. Each experiment involving one Gaussian state is repeated 1000 times to reduce statistical fluctuations. The process is carried out with ensembles of sizes 20, 10, 8 and 6. For a given ensemble size, the results for each member are averaged over 1000 runs and then the distance measures are averaged over the 100 states. We show that there is a clear advantage of using our scheme when the ensemble size is small. The test is carried out for three different sets of Gaussian states corresponding to three different temperatures given by  $\kappa = 1$ ,  $0.9$  and  $0.8$ , respectively.

The performance of state estimation of Gaussian states via our weak measurement protocol is compared to the corresponding performance of projective measurements. This is done via plots of the distance measures  $d_1$  and  $d_2$  vs weakness parameter defined by the inverse of squeezing  $\Delta q_m$  of the meter state, averaged over 100 such random states. The process is carried out for four different small ensemble sizes 20, 10, 8 and 6 and three different absolute temperatures given by  $\kappa = 1$ ,  $\kappa = 0.9$  and  $\kappa = 0.8$ .

Let us first look at Figure 3 (a). In this case the distance measures  $d_1$  and  $d_2$  have been plotted with  $\frac{1}{\Delta q_m}$

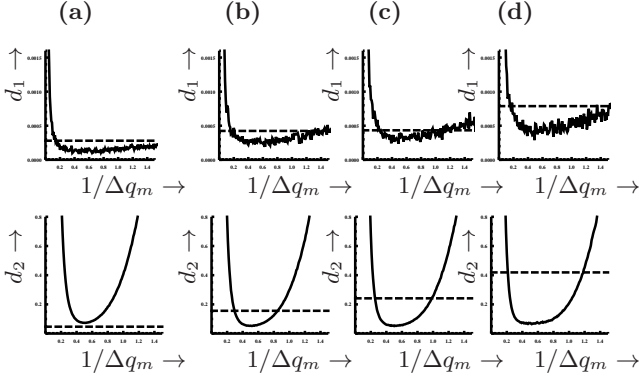


FIG. 4. The efficacy of our method as compared to projective measurements for  $\kappa = 0.9$  using averages over 100 random states further averaged over 1000 runs. The behaviors of  $d_1$  and  $d_2$  are plotted with  $1/\Delta q_m$  for ensemble sizes (a) 20 (b) 10 (c) 8 and (d) 6. The corresponding projective measurement results are plotted as dotted lines. While the method performs well in estimating the position of the Gaussian states for all ensemble sizes, as represented by  $d_1$ , it provides a clear advantage for estimating the spreads represented by  $d_2$  over projective measurements in the case of a small ensemble of size 6.

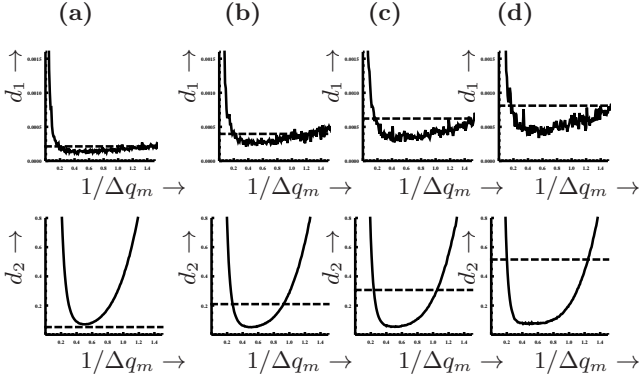


FIG. 5. The efficacy of our method as compared to projective measurements for  $\kappa = 0.8$  using averages over 100 random states further averaged over 1000 runs. The behaviors of  $d_1$  and  $d_2$  are plotted with  $1/\Delta q_m$  for ensemble sizes (a) 20 (b) 10 (c) 8 and (d) 6. The corresponding projective measurement results are plotted as dotted lines. While the method performs well in estimating the position of the Gaussian states for all ensemble sizes, as represented by  $d_1$ , it provides a clear advantage for estimating the spreads represented by  $d_2$  over projective measurements in the case of a small ensemble of size 6.

for an absolute temperature given by  $\kappa = 1$  and ensemble size 20. A low value of  $\frac{1}{\Delta q_m}$  indicates the meter prepared as a wide Gaussian in the position space. This corresponds to the weak measurement limit. A very weak measurement introduces a large amount of error in the measurement and this leads to a low quality of state estimation. This can be seen from the fact that the values of both  $d_1$  and  $d_2$ , on the left hand side of the plot for the

weak measurement method are much higher than those involving only projective measurements represented by the dotted line. Similarly, on the right side of the plot, the meter is prepared as a narrow Gaussian. The corresponding measurement limit for this side of the plot is that of strong projective measurements. Projective measurements destroy the state of the system and hence using the state for the second time leads to a low quality of state estimation. Only for an intermediate value of weakness, our method performs better than projective measurements. This is seen from the plot of  $d_1$  going below the dotted line representing the same distance measure for the projective measurement. The plot of  $d_2$  attains its minimum for the intermediate values of  $\frac{1}{\Delta q_m}$  but remains above the dotted line. It indicates that though our method has worked in giving a better estimation of the position of the Gaussian state, it does not perform as well to provide an estimation of the spreads of the Gaussian, in this particular case.

Figure 3 (b) shows the plot of the same parameters for the same absolute temperature but for a lower ensemble size of 10. We find that here our method proves to be more effective than the projective measurements both for the estimations of the position and the spread of the Gaussian Wigner function. Moving on to Figure 3(c) and (d) which are for the ensembles of sizes 8 and 6 we find that the relative efficacy of the estimation for position as well as the spread improves.

We repeat the same exercise with Gaussian states with finite temperatures with  $\kappa = 0.9$  and  $\kappa = 0.8$  as indicated in Figures 4 and 5, respectively. We observe the same trend as observed for the zero temperature in all these cases. Our method is not too effective in the extremely weak or extremely strong regimes. It works in the intermediate regimes depending upon the size of the ensemble and its efficacy increases with the lowering of the ensemble size.

In each of the plots, it is observed that the distance measures attain small values for an optimal value of squeezing. This is expected, as a very large value of squeezing ushers in too many errors into the “weak measurement”, while a small value causes a larger disturbance to the original state.

We observe from Figures 3, 4 and 5 that for an optimal range of  $1/\Delta q_m$  values, the weak distance measure curves go below the projective measurement line (represented by broken straight lines). In this regime of  $1/\Delta q_m$  values, our method is more effective than the projective measurement state estimation. The advantage is greater for smaller ensemble size. In fact for the ensemble size of 20 and  $\kappa = 1$ , the performances of the optimal weak measurement method and projective measurement are almost equal as can be seen in Figure 3. However, as the ensemble size decreases, a clear advantage emerges for the proposed scheme. There is no particular change in the advantage of our scheme relative to projective measurements, with change of temperature as is evident from plots with different temperature parameter  $\kappa$ .

## V. CONCLUDING REMARKS

In this paper, we have described our work on the estimation of Gaussian states by a method employing weak or unsharp measurements. We use phase space methods and the language of Wigner distributions for state estimation. We compare our results with state estimation based on projective measurements and show how one can do better in certain parameter regimes. Recycling of states, where one makes more than one measurement on a single copy before discarding it and tenability of the strength of the weak measurement are the two main ingredients of our scheme. The strength of the measurement is directly related to the amount of squeezing in the initial pointer state and can be tuned at will and we optimize the performance of our scheme with respect to this weakness parameter. The efficacy of the scheme is tested over a randomly chosen subset of Gaussian states. We demonstrate that the weak measurement based scheme produces a Wigner distribution which is much closer to the original Wigner distribution as com-

pared to the scheme based on projective measurements, for small ensemble sizes. As the ensemble size increases, the relative advantage of our scheme decreases, as seen in the comparative results for varying ensemble sizes. The behavior is repeated over the range of temperatures we have considered.

While in this work we have dealt with Gaussian states with the maximum spread along the  $q$  or  $p$  axes it will be interesting to extend the scheme to general Gaussian and non-Gaussian states. Another interesting direction that we are following up is to compare our results with schemes similar to the Arthurs and Kelly setup where position and momentum are measured together.

## ACKNOWLEDGMENTS

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